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MODELLING AND FORMULATION OF EQUATIONS OF MOTION FOR CRACKED ROTATING SHAFTS

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Abstract—The dynamics of a cracked, distributed parameter rotor component is studied. The proposed model is a rotating Timoshenko shaft which is also flexible in extension and torsion. Two uniform fields are connected by a local, complex spring element with reduced stiffness and damping simulating an open crack. The modification describing a closed crack is introduced, and the open/close condition is formulated. The governing boundary value problem is then derived. Finally, a special approach prepares the obtained set of equations for application of direct variational methods in a simplified form replacing the geometric discontinuity by a load discontinuity at the crack location. A simple example, discussing the torsional vibrations of a simply supported, circular rotating shaft with a circumferential crack, emphasizes the general considerations.

1. INTRODUCTION

Since the early 1970s when investigations on the vibrational behavior of cracked rotors began, numerous papers on this subject have been published, as a literature survey by Wauer (1990) shows. Today it can be stated that many problems have been solved and for the most important problem, the detection of cracks at an early stage, certain partial successes have been achieved.

In particular, the simplest model of the so-called deLaval-rotor has been treated in detail in papers by Gasch (1976), Henry and Okah-Avae (1976), Meyer (1979), Muszynska (1982) and Papadopoulos and Dimarogonas (1987). On the other hand, for distributed parameter rotor systems, one is under the impression that all authors pass quickly to real multi-shaft, multi-bearing turbine sets and purely numerical investigations, documented in papers by Grabowski (1980), Davies and Mayes (1984), Diana *et al.* (1986) and Nelson and Nataraj (1986). It can be supposed that some basic theoretical work would be helpful. Also, the special questions of damping of cracked vibrating structures and the most sensitive measures of vibration response to detect cracks seem not to have been discussed.

Following these ideas, the objective of this paper is to develop a beam-like rotating substructure in which the stiffness and damping properties of a single crack (or multiple ones) are more accurately modelled than previously. Subsequently, the governing equations of motion are derived. Replacing the two-field component with its geometric discontinuity by one uniform structure with a load discontinuity at the crack location, the governing boundary value problem is prepared to apply variational methods involving global shape functions. The set of equations obtained can be used as a starting point to find solutions approximated by formulas. Neglecting other influences, e.g. bearings, changes in the cross-sectional properties, or foundation, the investigations can then be focussed on the effect of the cracks. For a simple example, namely the torsional vibrations of a simply supported, circular shaft with a circumferential crack, the general procedure is illustrated.

Solutions of the generated equations of motion for more general cases and the search for an improved method to detect cracks will be kept in reserve for a future paper.

2. MODEL OF A CRACKED ROTOR ELEMENT AND GOVERNING BOUNDARY VALUE PROBLEM

As shown in Fig. 1, the idealized model of the rotor substructure used for this investigation consists of a uniform non-circular shaft of length l with a double-symmetric cross-section (see Fig. 2), mass per unit length μ, extensional rigidity EA (Young's modulus E,

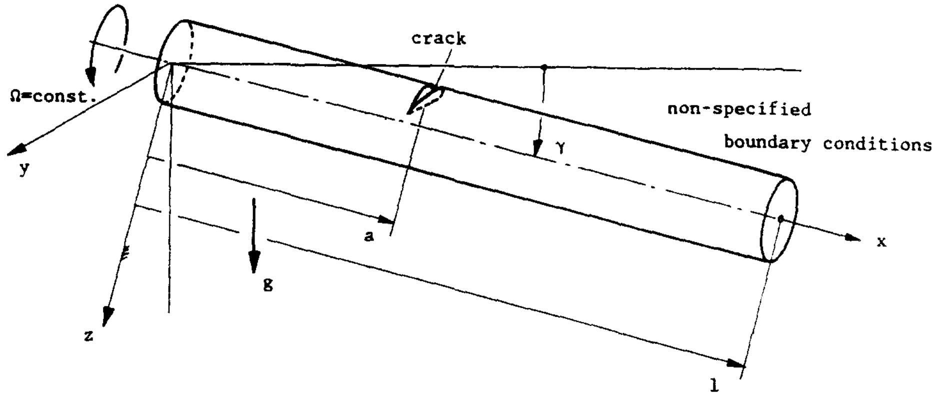

Fig. 1. Model of the cracked rotor substructure: 2-field non-circular Timoshenko-shaft with a double-symmetric cross section: EA, $\kappa_{1,2}$, $EI_{1,2}$, GI_T, μ, $r_{1,2}$, d_e, d_i.

cross-sectional area A), flexural rigidities $EI_{1,2}$ in the principal directions 1, 2 (sectional moments of inertia $I_{1,2}$), torsional stiffness GI_T (shear modulus G, torsional sectional moment of inertia I_T), shearing rigidities $\kappa_{1,2}GA$ (shape coefficient $\kappa_{1,2}$) and radii of gyration $r_{1,2}$. The shaft is assumed to be made of a viscoelastic material (damping coefficient d_i). Gyroscopic effects and the influence of shear on bending are taken into account. The shaft is to be exposed to external damping (damping coefficient d_e). The centroid W of the cross-sectional area is assumed to be different from the centre of mass S (eccentricity e, location angle δ), as shown in Fig. 2. The crack will be located at the position a $(0 < a < l)$ and is characterized by a stiffness matrix $K = \{k_{jk}\}$ and a damping matrix $D = \{d_{jk}\}$. A generalization to more than one crack is possible without fundamental difficulties. Instead of one crack location a with one governing stiffness and damping matrix K, D, there would be a finite number of them, a_i, K_i, D_i $(i = 1, 2, \ldots, n)$, where n is the number of cracks.

The coordinates of motion are the space- and time-dependent displacements $u(x, t)$ in the axial direction and $v(x, t)$, $w(x, t)$ in the transverse direction, the angles of inclination $\alpha(x, t)$ and $\beta(x, t)$ of the cross-section and the torsional deformation $\varphi(x, t)$. Subscripts t and x will denote partial derivatives with respect to time and space. The deformations are all measured in a body-fixed, rotating ξ, η, ζ reference frame, where the η- and ζ-axes are in the principal directions of the cross-section of the shaft and the ξ-axis coincides with the x-axis of a space-fixed, stationary x, y, z coordinate system (see Fig. 2). Assuming that the shaft rotates with a constant angular velocity Ω, the unit base vectors $\underline{e} = (\mathbf{e}_\xi, \mathbf{e}_\eta, \mathbf{e}_\zeta)^T$,

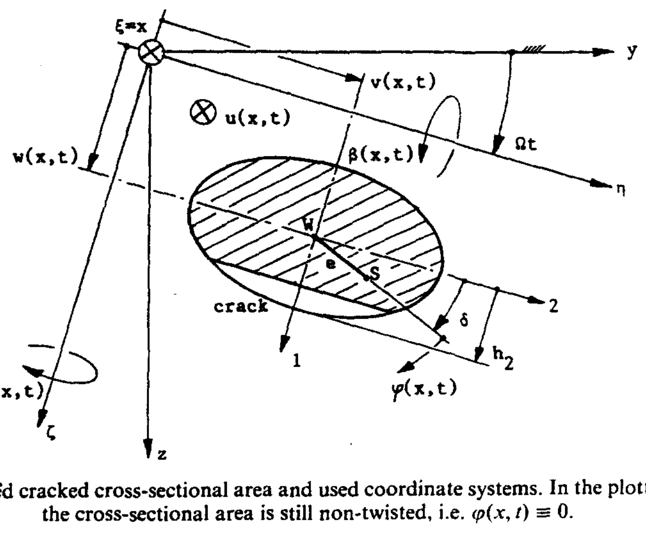

Fig. 2. Deformed cracked cross-sectional area and used coordinate systems. In the plotted position, the cross-sectional area is still non-twisted, i.e. $\varphi(x, t) \equiv 0$.

$\underline{i} = (\mathbf{i}_x, \mathbf{i}_y, \mathbf{i}_z)^T$ (the superscript T denotes transposition) of the coordinate systems introduced are connected by the transformation

$$\underline{e} = \underline{\Omega}\underline{i}, \quad \underline{\Omega} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \Omega t & \sin \Omega t \\ 0 & -\sin \Omega t & \cos \Omega t \end{bmatrix}. \tag{1}$$

To include both a horizontal and a vertical operation of the rotor considered, an angle γ is introduced which measures the slope between the x-axis and a horizontal plane. In the case $\gamma = 0$, for instance, the rotor lies in a horizontal position.

Originating with the well-established equations of motion for a rotating, circular, elastic Timoshenko shaft (see Eshleman and Eubanks, 1969, or Pedersen, 1972), the governing boundary value problem will be generated here, following Kelkel (1978) and Wauer (1983, 1986), by means of Hamilton's principle

$$\delta \int_{t_0}^{t_1} (T - V)\,\mathrm{d}t + \int_{t_0}^{t_1} \delta W\,\mathrm{d}t = 0 \tag{2}$$

where T is the kinetic energy, V is the elastic potential and δW is the virtual work of non-conservative forces for the rotating shaft.

The kinetic energy is given by (terms which give no contribution to the equations of motion and contain the eccentricity quadratically are omitted):

$$T = \int_0^l \frac{\mu}{2}\mathbf{v}^2\,\mathrm{d}x + \int_0^l \frac{\mu r_1^2}{2}(\alpha_t^2 - \Omega^2\alpha^2)\,\mathrm{d}x + \int_0^l \frac{\mu r_2^2}{2}(\beta_t^2 - \Omega^2\beta^2)\,\mathrm{d}x + \int_0^l \frac{\mu(r_1^2 + r_2^2)}{2}\varphi_t^2\,\mathrm{d}x, \tag{3}$$

where $\mathbf{v}$ is the absolute velocity of the cross-sectional mass centre S:

$$\mathbf{v} = u_t\mathbf{e}_\xi + \{v_t - e\varphi_t \sin(\delta + \varphi) - \Omega[w + e\sin(\delta + \varphi)]\}\mathbf{e}_\eta$$
$$+ \{w_t + e\varphi_t\cos(\delta + \varphi) + \Omega[v + e\cos(\delta + \varphi)]\}\mathbf{e}_\zeta. \tag{4}$$

The potential is a combination of the well-known potentials of the (here two) uncracked regions of the Timoshenko shaft

$$\begin{aligned} V_1 = {} & \int_0^{a_-} \frac{EI_1}{2}\alpha_x^2\,\mathrm{d}x + \int_{a_+}^l \frac{EI_1}{2}\alpha_x^2\,\mathrm{d}x + \int_0^{a_-} \frac{EI_2}{2}\beta_x^2\,\mathrm{d}x + \int_{a_+}^l \frac{EI_2}{2}\beta_x^2\,\mathrm{d}x \\ & + \int_0^{a_-} \frac{\kappa_1 GA}{2}(v_x - \alpha)^2\,\mathrm{d}x + \int_{a_+}^l \frac{\kappa_1 GA}{2}(v_x - \alpha)^2\,\mathrm{d}x + \int_0^{a_-} \frac{\kappa_2 GA}{2}(w_x + \beta)^2\,\mathrm{d}x \\ & + \int_{a_+}^l \frac{\kappa_2 GA}{2}(w_x + \beta)^2\,\mathrm{d}x + \int_0^{a_-} \frac{EA}{2}u_x^2\,\mathrm{d}x + \int_{a_+}^l \frac{EA}{2}u_x^2\,\mathrm{d}x + \int_0^{a_-} \frac{GI_T}{2}\varphi_x^2\,\mathrm{d}x \\ & + \int_{a_+}^l \frac{GI_T}{2}\varphi_x^2\,\mathrm{d}x, \end{aligned} \tag{5}$$

the gravity potential

$$V_2 = -\int_0^l \mu \mathbf{g}^T\mathbf{r}_S\,\mathrm{d}x \tag{6}$$

where $\mathbf{g}$ is the vector of gravity,

$$\mathbf{g} = g \sin \gamma \mathbf{i}_x + g \cos \gamma \mathbf{i}_z, \tag{7}$$

and $\mathbf{r}_S$ is the position vector of the cross-sectional mass centre,

$$\mathbf{r}_S = u\mathbf{e}_\xi + [v + e\cos(\delta+\varphi)]\mathbf{e}_\eta + [w + e\sin(\delta+\varphi)]\mathbf{e}_\zeta \tag{8}$$

and the potential of the cracked region (if the crack is open)

$$V_3 = \tfrac{1}{2}[\underline{u}(a_+, t) - \underline{u}(a_-, t)]^{\mathrm{T}} K [\underline{u}(a_+, t) - \underline{u}(a_-, t)] \tag{9}$$

where $\underline{u}$ is the column matrix of the different deformation variables

$$\underline{u} = (u, v, w, \alpha, \beta, \varphi)^{\mathrm{T}} \tag{10}$$

and K is the governing stiffness matrix (mentioned earlier). It can be noted that for a closed crack the deformations $\underline{u}(a_+, t)$ and $\underline{u}(a_-, t)$ will be identical, i.e. the part V_3 (9) vanishes and the stiffness properties $EI_{1,2}$, $\kappa_{1,2}GA$, EA and GI_T are continuous functions over the whole shaft region $0 < x < l$. It follows then that in eqn (5) there is only one interval of integration, from $x = 0$ to $x = l$, instead of two.

In the following, the body-fixed ξ, η, ζ coordinate system is assumed to be chosen in such a way that EI_2, $\kappa_2 GA$ are the smaller of the different shaft stiffnesses $EI_{1,2}$, $\kappa_{1,2}GA$, and the crack appears in the most unfortunate location (discussed by Muszynska, 1982), namely, that the crack edge is parallel to the η-axis (see Fig. 2). Then, for symmetry reasons (the 1-axis is a symmetry axis also for the cracked structure), the bending in the ζ-direction (w, β) and the longitudinal deformation (u) will be decoupled from the bending in the η-direction (v, α) and the torsional deformation (φ), as noticed by Gudmundson (1983). Thus, in contrast to assumptions made by Papadopoulos and Dimarogonas (1987a,b), the determination of the stiffness matrix K (and also the damping matrix D) can be divided into two decoupled problems. One obtains

$$K = \begin{bmatrix} k_{11} & 0 & 0 & 0 & k_{15} & 0 \\ 0 & k_{22} & 0 & 0 & 0 & k_{26} \\ 0 & 0 & k_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & k_{44} & 0 & 0 \\ k_{51} & 0 & 0 & 0 & k_{55} & 0 \\ 0 & k_{62} & 0 & 0 & 0 & k_{66} \end{bmatrix} \tag{11}$$

where, based on Irwin's (1960) classical work, the elements k_{jk} $(j, k = 1, \ldots, 6)$ can be approximated by formulas as shown in different papers by Papadopoulos and Dimarogonas (1987a,b), Gudmundson (1983) and Haisty *et al.* (1988) or numerically calculated by finite element methods, for instance, as discussed by Gudmundson (1983) and Schmalhorst (1988). It is noted that in analogy to the potential V_1 (5), perhaps a stiffness matrix K (11) would be more consistent, if it would have non-vanishing elements $k_{24} = k_{42}$ and $k_{35} = k_{53}$.

Finally, the virtual work has to be added. Assuming that the only non-conservative forces are damping forces, according to a paper by Wauer (1985) the virtual work is composed of the virtual work of external damping

$$\delta W_1 = -d_e \int_0^l \mu[u_t \delta u + (v_t - \Omega w)\delta v + (w_t + \Omega v)\delta w + r_1^2 \alpha_t \delta\alpha + r_2^2 \beta_t \delta\beta + (r_1^2 + r_2^2)\varphi_t \delta\varphi]\, \mathrm{d}x, \tag{12}$$

the virtual work of the viscoelastic shaft sections

$$\delta W_2 = -d_i\left[\int_0^{a_-} EAu_{,tx}\delta u_x\,\mathrm{d}x + \int_{a_+}^{l} EAu_{,tx}\delta u_x\,\mathrm{d}x + \int_0^{a_-} EI_1\alpha_{,tx}\delta\alpha_x\,\mathrm{d}x + \int_{a_+}^{l} EI_1\alpha_{,tx}\delta\alpha_x\,\mathrm{d}x\right.$$

$$+\int_0^{a_-} EI_2\beta_{,tx}\delta\beta_x\,\mathrm{d}x + \int_{a_+}^{l} EI_2\beta_{,tx}\delta\beta_x\,\mathrm{d}x + \int_0^{a_-} \kappa_1 GA(v_{,tx}-\alpha_{,t})\delta(v_x-\alpha)\,\mathrm{d}x$$

$$+\int_{a_+}^{l} \kappa_1 GA(v_{,tx}-\alpha_{,t})\delta(v_x-\alpha)\,\mathrm{d}x + \int_0^{a_-} \kappa_2 GA(w_{,tx}+\beta_{,t})\delta(w_x+\beta)\,\mathrm{d}x$$

$$\left.+\int_{a_+}^{l} \kappa_2 GA(w_{,tx}+\beta_{,t})\delta(w_x+\beta)\,\mathrm{d}x + \int_0^{a_-} GI_T\varphi_{,tx}\delta\varphi_x\,\mathrm{d}x + \int_{a_+}^{l} GI_T\varphi_{,tx}\delta\varphi_x\,\mathrm{d}x\right] \quad (13)$$

and, in analogy to the stiffness potential V_3 (9), the virtual work of the damaged region

$$\delta W_3 = [\underline{u}_{,t}(a_+,t)-\underline{u}_{,t}(a_-,t)]^{\mathrm{T}} D[\delta\underline{u}(a_+,t)-\delta\underline{u}(a_-,t)] \quad (14)$$

where

$$D = \begin{bmatrix} d_{11} & 0 & 0 & 0 & d_{15} & 0 \\ 0 & d_{22} & 0 & 0 & 0 & d_{26} \\ 0 & 0 & d_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & d_{44} & 0 & 0 \\ d_{51} & 0 & 0 & 0 & d_{55} & 0 \\ 0 & d_{62} & 0 & 0 & 0 & d_{66} \end{bmatrix} \quad (15)$$

For an open crack, the elements d_{jk} are assumed to be proportional to the corresponding stiffness properties, i.e.

$$d_{jk} = d_i k_{jk}, \quad j,k = 1,2,\ldots,6. \quad (16)$$

For a closed crack, a significant modification of the considerations formulating the stiffness properties is proposed. It is assumed that the virtual work δW_3 (14) vanishes again $[\underline{u}_{,t}(a_+,t) \equiv \underline{u}_{,t}(a_-,t)]$ and in the part δW_2 (13) a continuous interval of integration appears, but taking into account that in cracked structures the damping may increase (Cawley and Adams, 1979; Schmalhorst, 1988, private communication) and this fact can be explained in that the compressed crack faces induce dry friction, so an additional damping is introduced. For simplicity, in δW_1 (7) an additional term

$$\delta W_4 = -k_C\mu[u_{,t}\delta u + (v_{,t}-\Omega w)\delta v + (w_{,t}+\Omega v)\delta w + r_1^2\alpha_{,t}\delta\alpha + r_2^2\beta_{,t}\delta\beta + (r_1^2+r_2^2)\varphi_{,t}\delta\varphi]_{x=a} \quad (17)$$

is considered in the form of external damping concentrated at the crack location.

The last step in the model description is the formulation of the closing conditions of the "breathing" crack. For simplification, partially open cracks are not admitted; the crack is either completely open or completely closed. A physically plausible condition used here is that for an open crack, for instance, the elongation of every point on the crack face must be positive. If h_2 is the smaller of the "radii" of the oval cross-sectional shaft area, the open/close relations can then be defined as follows:

$$(u_x - h_2\beta_x)|_{x=a} > 0 \quad \text{for an open crack,}$$
$$(u_x - h_2\beta_x)|_{x=a} < 0 \quad \text{for a closed crack.} \quad (18)$$

Now the governing field equations for a rotating Timoshenko shaft can be derived. Carrying out the required variations in Hamilton's principle (2) and using all formulated relationships (3)–(17), the linearized equations of motion for the two uncracked shaft sections are

$$
\begin{aligned}
&\mu u_{tt}+d_e\mu u_t-EA(u_{xx}+d_i u_{txx})-\mu g\sin\gamma=0,\\
&\mu(v_{tt}-2\Omega w_t-\Omega^2 v)+d_e\mu(v_t-\Omega w)-\kappa_1 GA[v_{xx}-\alpha_x+d_i(v_{txx}-\alpha_{tx})]\\
&\qquad -e\mu[(\Omega^2+2\Omega\varphi_t)\cos\delta+(\varphi_{tt}-\Omega^2\varphi)\sin\delta]-\mu g\cos\gamma\sin\Omega t=0,\\
&\mu(w_{tt}+2\Omega v_t-\Omega^2 w)+d_e\mu(w_t+\Omega v)-\kappa_2 GA[w_{xx}+\beta_x+d_i(w_{txx}+\beta_{tx})]\\
&\qquad -e\mu[(\Omega^2+2\Omega\varphi_t)\sin\delta-(\varphi_{tt}-\Omega^2\varphi)\cos\delta]-\mu g\cos\gamma\cos\Omega t=0,\\
&\mu r_1^2(\alpha_{tt}+\Omega^2\alpha)+d_e\mu r_1^2\alpha_t-EI_1(\alpha_{xx}+d_i\alpha_{txx})-\kappa_1 GA[v_x-\alpha+d_i(v_{tx}-\alpha_t)]=0,\\
&\mu r_2^2(\beta_{tt}+\Omega^2\beta)+d_e\mu r_2^2\beta_t-EI_2(\beta_{xx}+d_i\beta_{txx})+\kappa_2 GA[w_x+\beta+d_i(w_{tx}+\beta_t)]=0,\\
&\mu(r_1^2+r_2^2)\varphi_{tt}+d_e\mu(r_1^2+r_2^2)\varphi_t-GI_T(\varphi_{xx}+d_i\varphi_{txx})\\
&\qquad -e\mu[(v_{tt}-2\Omega w_t-\Omega^2 v)\sin\delta-(w_{tt}+2\Omega v_t-\Omega^2 w)\cos\delta]\\
&\qquad -\mu e g\cos\gamma[-\sin\delta\sin\Omega t+\cos\delta\cos\Omega t]=0,\\
&\qquad 0<x<a_- \quad\text{and}\quad a_+<x<l.
\end{aligned}
\tag{19}
$$

The corresponding boundary conditions at the outside boundaries of the two fields need not be specified at this moment. As transition conditions at the damaged region, one obtains (for an open crack)

$$
\begin{aligned}
\underline{P}(a_-,t)&=\underline{P}(a_+,t),\\
\underline{P}(a_+,t)&=K[\Delta\underline{u}(a,t)+d_i\Delta\underline{u}_t(a,t)]
\end{aligned}
\tag{20}
$$

where

$$
\begin{aligned}
\underline{P}=\{&EA(u_x+d_iu_{tx}),\quad \kappa_1GA[v_x-\alpha+d_i(v_{tx}-\alpha_t)],\quad \kappa_2GA[w_x+\beta+d_i(w_{tx}+\beta_t)],\\
&EI_1(\alpha_x+d_i\alpha_{tx}),\quad EI_2(\beta_x+d_i\beta_{tx}),\quad GI_T(\varphi_x+d_i\varphi_{tx})\}^{\mathrm{T}},\\
\Delta\underline{u}(a,t)&=\underline{u}(a_+,t)-\underline{u}(a_-,t),\quad \Delta\underline{u}_t(a,t)=\underline{u}_t(a_+,t)-\underline{u}_t(a_-,t).
\end{aligned}
\tag{21}
$$

For a closed crack, the transition conditions (20) drop out and, in the field eqns (19), which are now continuously valid in the whole region $0<x<l$, the damping coefficient d_e has to be replaced by $d_e+d_C\delta(x-a)$, where $\delta(x)$ is the delta function.

The boundary value problem derived (including boundary conditions) can be considered as the generalization of the equations of motion of a cracked deLaval-rotor studied by Gasch (1976), Henry and Okah-Avae (1976), Meyer (1979), Muszynska (1982) and Papadopoulos and Dimarogonas (1987a) to a distributed parameter system, formulating the governing relations in body-fixed, rotating coordinates. Several special cases can be recognized. For a non-rotating Timoshenko shaft with an open crack, treating only the free coupled bending/torsional vibrations $v(x,t)$, $\alpha(x,t)$ and $\varphi(x,t)$, the boundary value problem reduces to that given by Papadopoulos and Dimarogonas (1987b). The remaining boundary value problem describing coupled longitudinal/bending vibrations $u(x,t)$ and $w(x,t)$, $\beta(x,t)$ is a slight generalization of that in another paper by Papadopoulos and Dimarogonas (1988). The simplest case of a cracked slender structure can be obtained by a further restriction. If the crack geometry (during deformation) is assumed to be symmetric in the form of a double-sided crack (for an oval shaft) or a circumferential one (for a circular cross-section), the extensional vibrations $u(x,t)$ and the bending oscillations $w(x,t)$, $\beta(x,t)$ decouple ($k_{15}, k_{51}, d_{15}, d_{51}\equiv 0$). The resulting boundary value problem for $u(x,t)$ consists of a wave equation for each of the two sections coupled by the transition conditions at the crack position. Finally, for an uncracked rotating Timoshenko shaft, the equations of motion go over to those derived, for instance, by Wauer (1983, 1986).

For slender shafts, one can go over by a well-known limiting process, discussed in detail by Kelkel (1978), from the Timoshenko theory to the Bernoulli/Euler theory. The field eqns (19) change into

$$\begin{gathered}\mu u_{tt}+d_e\mu u_t-EA(u_{xx}+d_iu_{txx})-\mu g\sin\gamma=0,\\ \mu(v_{tt}-2\Omega w_t-\Omega^2v)+d_e\mu(v_t-\Omega w)+EI_1(v_{xxxx}+d_iv_{txxxx})\\ -e\mu[(\Omega^2+2\Omega\varphi_t)\cos\delta+(\varphi_{tt}-\Omega^2\varphi)\sin\delta]-\mu g\cos\gamma\sin\Omega t=0,\\ \mu(w_{tt}+2\Omega v_t-\Omega^2w)+d_e\mu(w_t+\Omega v)+EI_2(w_{xxxx}+d_iw_{txxxx})\\ -e\mu[(\Omega^2+2\Omega\varphi_t)\sin\delta-(\varphi_{tt}-\Omega^2\varphi)\cos\delta]-\mu g\cos\gamma\cos\Omega t=0,\\ \mu(r_1^2+r_2^2)\varphi_{tt}+d_e\mu(r_1^2+r_2^2)\varphi_t-GI_T(\varphi_{xx}+d_i\varphi_{txx})\\ -e\mu[(v_{tt}-2\Omega w_t-\Omega^2v)\sin\delta-(w_{tt}+2\Omega v_t-\Omega^2w)\cos\delta]\\ -\mu eg\cos\gamma[-\sin\delta\sin\Omega t+\cos\delta\cos\Omega t]=0,\\ 0<x<a_-\quad\text{and}\quad a_+<x<l\end{gathered}\tag{22}$$

and the corresponding transition conditions (20) are (for an open crack)

$$\begin{aligned}\underline{Q}(a_-,t)&=\underline{Q}(a_+,t),\\ \underline{Q}(a_+,t)&=K[\Delta\underline{v}(a,t)+d_i\Delta\underline{v}_t(a,t)]\end{aligned}\tag{23}$$

where

$$\underline{Q}=[EA(u_x+d_iu_{tx}),\quad EI_1(v_{xxx}+d_iv_{txxx}),\quad EI_2(w_{xxx}+d_iw_{txxx}),\quad EI_1(v_{xx}+d_iv_{txx}),\\ EI_2(w_{xx}+d_iw_{txx}),\quad GI_T(\varphi_x+d_i\varphi_{tx})]^{\mathrm{T}}.\tag{24}$$

It is obvious that in the state vector (10) two elements now depend on two other ones, and it modifies to

$$\underline{v}=(u,v,w,v_x,w_x,\varphi)^{\mathrm{T}}.\tag{25}$$

Since in a Bernoulli/Euler beam model shear deformation is neglected, the stiffness elements $k_{22}, k_{33}, k_{26}=k_{62}$ have to be chosen infinitely large, so that as a consequence the transverse displacements v, w are continuous at the crack location (mentioned by Papadopoulos and Dimarogonas, 1988).

3. APPROXIMATE EQUATIONS OF MOTION

The aim of this section is to prepare the field equations and transition conditions of the cracked shaft component, eqns (19) to (21), in such a way that solutions by formulae can be found, e.g. by Galerkin procedures. However, to take into account the crack effect appearing in the transition conditions (20), two-field solutions with different values at $x=a_-$ and $x=a_+$, even in the case $a_+-a_-=\varepsilon\to 0$, have to be formulated. To avoid this complication, a re-writing of the equations of motion will be undertaken. Following an idea of Thomson (1949) and Kirmser (1944), taken up by Petroski (1981) and Chang and Petroski (1986), the "new" model equations replace the geometric discontinuity (appearing as a two-field beam connected by a generalized viscoelastic spring element) by a load discontinuity at the crack location. Thus, effectively, a local reduction of stiffness and material damping (for an open crack) is introduced into an otherwise uniform shaft. It follows that an approximate solution using global shape functions will be possible.

To establish these ideas, first in the direct neighborhood of the crack location, i.e. $x = a_-$ and $x = a_+$, a couple of generalized loads (see Fig. 3)

$$\underline{F} = (F_x, F_\eta, F_\zeta, M_\zeta, M_\eta, M_x)^T \tag{26}$$

and

$$\underline{f} = (f_x, f_\eta, f_\zeta, m_\zeta, m_\eta, m_x)^T \tag{27}$$

are introduced to locally reduce the shaft stiffness and damping, respectively. An elementary static calculation leads to the corresponding deformation as defined in eqn (10),

$$\Delta\underline{u}(a) = \varepsilon S\underline{F} \tag{28}$$

where (for $\varepsilon \to 0$)

$$S = \begin{bmatrix} 1/EA & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/\kappa_1 GA & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/\kappa_2 GA & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/EI_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/EI_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/GI_T \end{bmatrix} \tag{29}$$

and, assuming a viscous connecting link can be applied in an analogous manner, also its time derivative

$$\Delta\underline{u}_t(a) = \varepsilon \frac{S}{d_i}\underline{f} \tag{30}$$

as a function of loads $\underline{F}$ (26) and $\underline{f}$ (27), respectively. From eqns (28) and (30) it follows that

$$\Delta\underline{u}(a) + d_i \Delta\underline{u}_t(a) = \varepsilon S(\underline{F} + \underline{f}). \tag{31}$$

From a comparison of relation (31) with the transition conditions (20), a correlation between the stiffness matrix K (11) and the damping coefficient d_i, characterizing the initial geometric discontinuity and the loads $\underline{F}$, $\underline{f}$ (including the location property ε), can be

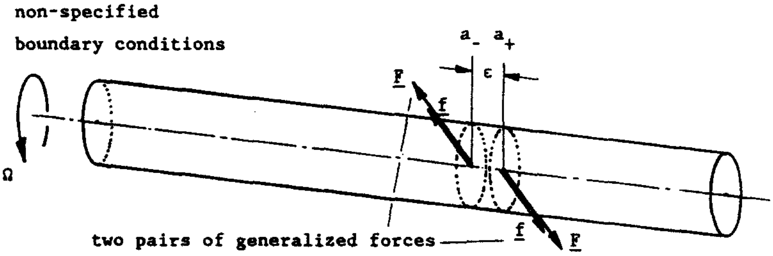

Fig. 3. Shaft component with a load discontinuity: 1-field uniform shaft: EA, $\kappa_{1,2}$, $EI_{1,2}$, GI_T, μ, $r_{1,2}$, d_e, d_i.

formulated. From the presupposed equivalence of the displacements and velocities in the relations (20) and (31),

$$\varepsilon S(\underline{F}+\underline{f}) = K^{-1}\underline{P}(a_+) \tag{32}$$

can be concluded. The inverse stiffness matrix K^{-1} is the so-called compliance matrix B and its elements b_{jk} are available from the literature, even more so than the elements k_{jk} of the stiffness matrix K (11).

The original two-field formulation, eqns (19)–(21), for an open crack can now be replaced by the new equations of motion

$$\begin{aligned}
&\mu u_{tt}+d_e\mu u_t-EA(u_{xx}+d_iu_{txx}) = \mu g\sin\gamma+(F_x+f_x)[\delta(x-a_+)-\delta(x-a_-)],\\
&\mu(v_{tt}-2\Omega w_t-\Omega^2 v)+d_e\mu(v_t-\Omega w)-\kappa_1 GA[v_{xx}-\alpha_x+d_i(v_{txx}-\alpha_{tx})]\\
&\qquad = e\mu[(\Omega^2+2\Omega\varphi_t)\cos\delta+(\varphi_{tt}-\Omega^2\varphi)\sin\delta]+\mu g\cos\gamma\sin\Omega t\\
&\qquad\qquad +(F_\eta+f_\eta)[\delta(x-a_+)-\delta(x-a_-)],\\
&\mu(w_{tt}+2\Omega v_t-\Omega^2 w)+d_e\mu(w_t+\Omega v)-\kappa_2 GA[w_{xx}+\beta_x+d_i(w_{txx}+\beta_{tx})]\\
&\qquad = e\mu[(\Omega^2+2\Omega\varphi_t)\sin\delta-(\varphi_{tt}-\Omega^2\varphi)\cos\delta]+\mu g\cos\gamma\cos\Omega t\\
&\qquad\qquad +(F_\zeta+f_\zeta)[\delta(x-a_+)-\delta(x-a_-)],\\
&\mu r_1^2(\alpha_{tt}+\Omega^2\alpha)+d_e\mu r_1^2\alpha_t-EI_1(\alpha_{xx}+d_i\alpha_{txx})-\kappa_1 GA[v_x-\alpha+d_i(v_{tx}-\alpha_t)]\\
&\qquad\qquad = (M_\zeta+m_\zeta)[\delta(x-a_+)-\delta(x-a_-)],\\
&\mu r_2^2(\beta_{tt}+\Omega^2\beta)+d_e\mu r_2^2\beta_t-EI_2(\beta_{xx}+d_i\beta_{txx})+\kappa_2 GA[w_x+\beta+d_i(w_{tx}+\beta_t)]\\
&\qquad\qquad = (M_\eta+m_\eta)[\delta(x-a_+)-\delta(x-a_-)],\\
&\mu(r_1^2+r_2^2)\varphi_{tt}+d_e\mu(r_1^2+r_2^2)\varphi_t-GI_T(\varphi_{xx}+d_i\varphi_{txx})\\
&\qquad = e\mu[(v_{tt}-2\Omega w_t-\Omega^2 v)\sin\delta-(w_{tt}+2\Omega v_t-\Omega^2 w)\cos\delta]\\
&\qquad -\mu eg\cos\gamma[\sin\delta\sin\Omega t-\cos\delta\cos\Omega t]+(M_x+m_x)[\delta(x-a_+)-\delta(x-a_-)],\\
&\qquad\qquad 0<x<l.
\end{aligned} \tag{33}$$

The transition conditions (20) no longer exist explicitly; they appear in the form of locally concentrated generalized forces $\underline{F}$, $\underline{f}$ [using Dirac's delta function $\delta(x)$] in relations (33). The advantage of the proposed idea is obvious: the number of field equations has been halved. While originally there were two fields with six partial differential equations for each segment coupled by complicated transition conditions, now the shaft is one uniform section with a more complicated "external" load configuration.

In an additional step, the forces $(\underline{F}+\underline{f})$ on the right-hand side of eqns (33) can be replaced, using relation (32), by the inverse matrix S^{-1} multiplied by the product $\underline{K}^{-1}\underline{P}(a_+)/\varepsilon$. However, this will only be executed for a special example, see the next section.

For a closed crack, instead of the transition conditions (20) the terms $(\underline{F}+\underline{f})$ now have to be omitted. The modification of the damping coefficient d_e remains the same as described earlier.

The procedure introduced will finally be applied to a Bernoulli/Euler shaft. The load vectors (26) and (27) shorten to

$$\underline{H} = (F_x, M_\zeta, M_\eta, M_x)^{\mathrm{T}} \tag{34}$$

and

$$\underline{h} = (f_x, m_\zeta, m_\eta, m_x)^T. \tag{35}$$

respectively. Relation (31) goes over into

$$\Delta \underline{v}_1(a) + d_i \Delta \underline{v}_{1t}(a) = \varepsilon R(\underline{H} + \underline{h}) \tag{36}$$

where

$$\underline{v}_1 = (u, v_x, w_x, \varphi)^T, \quad \Delta \underline{v}_1(a) = \underline{v}_1(a_+) - \underline{v}_1(a_-), \quad \Delta \underline{v}_{1t}(a) = \underline{v}_{1t}(a_+) - \underline{v}_{1t}(a_-) \tag{37}$$

and

$$R = \begin{bmatrix} 1/EA & 0 & 0 & 0 \\ 0 & 1/EI_1 & 0 & 0 \\ 0 & 0 & -1/EI_2 & 0 \\ 0 & 0 & 0 & 1/GI_T \end{bmatrix}. \tag{38}$$

Instead of eqn (32) one obtains

$$\varepsilon R(\underline{H} + \underline{h}) = K_1^{-1} \underline{Q}_1(a_-) \tag{39}$$

where $\underline{Q}_1$ and K_1 are constructed from $\underline{Q}$ (24) and K (11) omitting the second and third line and second and third line and row, respectively.

The model equations for a Bernoulli/Euler shaft with an open crack then read

$$\begin{aligned}
&\mu u_{tt} + d_e \mu u_t - EA(u_{xx} + d_i u_{txx}) = \mu g \sin\gamma + (F_x + f_x)[\delta(x - a_+) - \delta(x - a_-)], \\
&\mu(v_{tt} - 2\Omega w_t - \Omega^2 v) + d_e \mu(v_t - \Omega w) + EI_1(v_{xxxx} + d_i v_{txxxx}) \\
&\qquad = e\mu[(\Omega^2 + 2\Omega\varphi_t)\cos\delta + (\varphi_{tt} - \Omega^2\varphi)\sin\delta] + \mu g \cos\gamma \sin\Omega t \\
&\qquad\qquad - (M_\zeta + m_\zeta)[\delta^{(1)}(x - a_+) - \delta^{(1)}(x - a_-)], \\
&\mu(w_{tt} + 2\Omega v_t - \Omega^2 w) + d_e \mu(w_t + \Omega v) + EI_2(v_{xxxx} + d_i w_{txxxx}) \\
&\qquad = e\mu[(\Omega^2 + 2\Omega\varphi_t)\sin\delta - (\varphi_{tt} - \Omega^2\varphi)\cos\delta] + \mu g \cos\gamma \cos\Omega t \\
&\qquad\qquad + (M_\eta + m_\eta)[\delta^{(1)}(x - a_+) - \delta^{(1)}(x - a_-)], \\
&\mu(r_1^2 + r_2^2)\varphi_{tt} + d_e \mu(r_1^2 + r_2^2)\varphi_t - GI_T(\varphi_{xx} + d_i \varphi_{txx}) \\
&\qquad = e\mu[(v_{tt} - 2\Omega w_t - \Omega^2 v)\sin\delta - (w_{tt} + 2\Omega v_t - \Omega^2 w)\cos\delta] \\
&\qquad - \mu e g \cos\gamma[\sin\delta \sin\Omega t - \cos\delta \cos\Omega t] + (M_x + m_x)[\delta(x - a_+) - \delta(x - a_-)], \\
&\qquad\qquad 0 < x < l.
\end{aligned} \tag{40}$$

where the superscript enclosed in brackets denotes the derivative of the δ-function.

It will be noted that in formulae (33) and (40) it is formally simple to include additional rigid disks (masses m_i, moments of inertia with respect to the ξ, η, ζ axes $J_i^{\xi,\eta,\zeta}$, $i = 1, \ldots, N$, where N is the number of disks). The initially uniform mass properties μ, $\mu r_{1,2}^2$, $\mu(r_1^2 + r_2^2)$ are then to be replaced by

$$\begin{aligned}
\mu^* &= \mu + m_i \delta(x - x_i), \\
(\mu r_1^2)^* &= \mu r_1^2 + J_i^\eta \delta(x - x_i), \quad (\mu r_2^2)^* = \mu r_2^2 + J_i^\zeta \delta(x - x_i), \\
[\mu(r_1^2 + r_2^2)]^* &= \mu(r_1^2 + r_2^2) + J_i^\xi \delta(x - x_i),
\end{aligned} \tag{41}$$

where the x_i $(0 < x_i < l)$ denote the locations of the disks. A theoretical basis by which the formulation of parameters and also load discontinuities can be justified is described for the case of stepwise changes in a paper by Bernasconi (1986).

In the form (33) or (40), supplemented by appropriate boundary conditions, the equations of motion for a cracked, rotating shaft are prepared for the application of direct variational methods to obtain approximate solutions.

4. EXAMPLE

Illustrating the general procedure described in the previous chapters, and calculating a first simple result, pure torsional vibrations of a circular shaft (radius r) with a circumferential crack will be discussed. Since in this case a closed crack does not appear during deformation, the complete dynamic equations are represented by eqns (19)–(21), reducing to two single differential equations

$$\mu(r_1^2+r_2^2)(\varphi_{tt}+d_e\varphi_t)-GI_T(\varphi_{xx}+d_i\varphi_{txx}) = eg\cos\gamma(\cos\delta\cos\Omega t-\sin\delta\sin\Omega t),$$
$$0<x<a_- \quad \text{and} \quad a_+<x<l \tag{42}$$

with the transition conditions

$$GI_T(\varphi_x+d_i\varphi_{tx})|_{x=a_-} = GI_T(\varphi_x+d_i\varphi_{tx})|_{x=a_+},$$
$$GI_T(\varphi_x+d_i\varphi_{tx})|_{x=a_+} = k_{66}\{\Delta\varphi(a,t)+d_i\Delta\varphi_t(a,t)\}. \tag{43}$$

As boundary conditions those of a free–free shaft are assumed in the form

$$GI_T(\varphi_x+d_i\varphi_{tx})|_{x=0,l} = 0. \tag{44}$$

The procedure of Section 3 then leads to the approximate single field equation

$$\mu(r_1^2+r_2^2)(\varphi_{tt}+d_e\varphi_t)-GI_T(\varphi_{xx}+d_i\varphi_{txx}) = eg\cos\gamma(\cos\delta\cos\Omega t-\sin\delta\sin\Omega t)$$
$$+(M_x+m_x)\{\delta(x-a_+)-\delta(x-a_-)\},\quad 0<x<l \tag{45}$$

with unchanged boundary conditions (44). Subsequently, the load property (M_x+m_x) is replaced using eqn (32) so that the field eqn (45) is modified to

$$\mu(r_1^2+r_2^2)(\varphi_{tt}+d_e\varphi_t)-GI_T(\varphi_{xx}+d_i\varphi_{txx}) = eg\cos\gamma(\cos\delta\cos\Omega t-\sin\delta\sin\Omega t)$$
$$+(GI_T)^2[\varphi_x(a_+,t)+d_i\varphi_{tx}(a_+,t)][\delta(x-a_+)-\delta(x-a_-)]/(\varepsilon k_{66}),\quad 0<x<l. \tag{46}$$

In the next step, using global shape functions which satisfy all boundary conditions (44), let

$$\varphi(x,t) = \sum_m \cos m\pi\frac{x}{l}f_m(t). \tag{47}$$

In the usual manner, Galerkin's procedure yields a system of linear ordinary differential equations ($\dot{} = d/dt$)

$$\int_0^l\left\{\sum_m \mu(r_1^2+r_2^2)\cos m\pi\frac{x}{l}(\ddot{f}_m+d_e\dot{f}_m)+GI_T\left(\frac{m\pi}{l}\right)^2\cos m\pi\frac{x}{l}(f_m+d_i\dot{f}_m)\right\}\cdot\cos n\pi\frac{x}{l}\,dx$$
$$=\int_0^l eg\cos\gamma(\cos\delta\cos\Omega t-\sin\delta\sin\Omega t)\cos n\pi\frac{x}{l}\,dx$$

$$-\int_0^l \sum_m \frac{(GI_T)^2}{\varepsilon k_{66}} \left\{ \frac{m\pi}{l} \sin m\pi \frac{a}{l} (f_m + d_i \dot{f}_m) \cos n\pi \frac{x}{l} [\delta(x-a_+) - \delta(x-a_-)] \right\} \mathrm{d}x.$$

$$n = 0, 1, \ldots . \qquad (48)$$

Taking notice of the orthogonality conditions (δ_{mn} = Kronecker's delta)

$$\int_0^l \cos m\pi \frac{x}{l} \cos n\pi \frac{x}{l} \,\mathrm{d}x = \int_0^l \sin m\pi \frac{x}{l} \sin n\pi \frac{x}{l} \,\mathrm{d}x = \frac{1}{2} \delta_{mn} \qquad (49)$$

and the fact that for $\varepsilon \ll l$ the relation

$$\int_0^l \cos n\pi \frac{x}{l} \{\delta(x-a_+) - \delta(x-a_-)\} \,\mathrm{d}x = -\varepsilon \frac{n\pi}{l} \sin n\pi \frac{a}{l} \qquad (50)$$

is approximately valid, the system of differential equations (48) can be simplified to

$$\frac{\mu}{2}(r_1^2+r_2^2)l(\ddot{f}_n + d_e \dot{f}_n) + \frac{1}{2} GI_T \left(\frac{n\pi}{l}\right)^2 (f_n + d_i \dot{f}_n)$$

$$-(GI_T)^2 b_{66} \frac{n\pi}{l} \sin n\pi \frac{a}{l} \sum_m \frac{m\pi}{l} \sin m\pi \frac{a}{l} (f_m + d_i \dot{f}_m)$$

$$= eg \cos\gamma (\cos\delta \cos\Omega t - \sin\delta \sin\Omega t) l \quad \text{for } n = 0 \quad \text{and} \quad 0 \text{ for } n > 0. \qquad (51)$$

In a 1-term approximation—ignoring the rigid body motion—one ends up with a single differential equation

$$\frac{\mu}{2}(r_1^2+r_2^2)l(\ddot{f}_1 + d_e \dot{f}_1) + \frac{1}{2} GI_T \left(\frac{\pi}{l}\right)^2 (f_1 + d_i \dot{f}_1) - (GI_T)^2 b_{66} \left(\frac{\pi}{l}\right)^2 \cdot \sin^2 \frac{\pi}{l} a (f_1 + d_i \dot{f}_1) = 0.$$

$$(52)$$

Here, only the lowest natural frequency ω_1, and especially the frequency drop $\Delta\omega_1$ versus compliance b_{66} and crack location a, will be calculated. Straightforwardly, from eqn

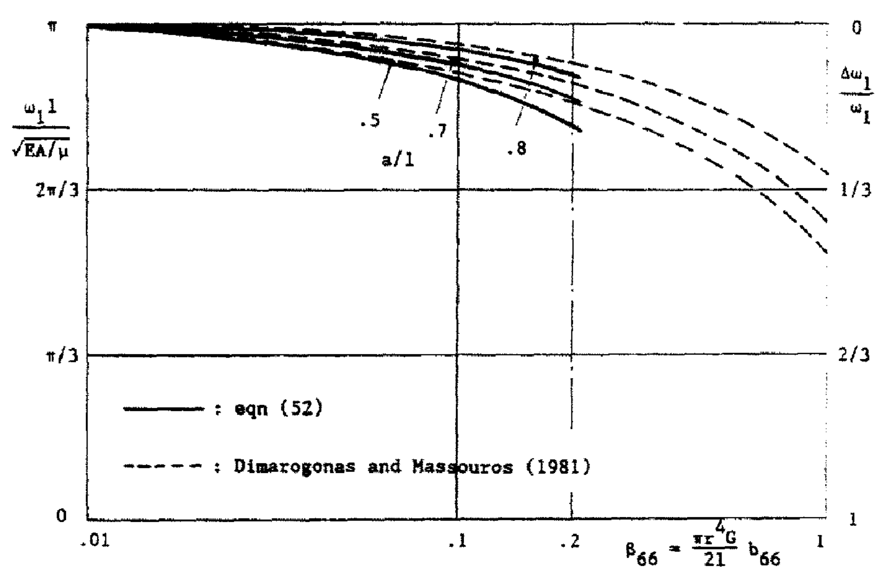

Fig. 4. Frequency drop for the lowest harmonic versus compliance (crack depth) for several crack locations.

(52) one concludes the result depicted in Fig. 4. The dependence of compliance on the crack depth can be taken from a paper by Dimarogonas and Massouros (1981), in which the natural frequency is calculated based on the two-field boundary value problem, eqns (42)–(44). Indeed, for this simple case, the corresponding eigenvalue problem can be solved exactly, but the calculation expense is considerable.

The comparison shows that for a sufficiently small compliance parameter $\beta_{66} \lesssim 0.2$ (appearing in the most practical applications) even a 1-term approximation, very easy to handle, leads to accurate results; the maximal differences are limited to 5%. Only for deeper cracks does this lowest-order approximation become useless and then a higher calculation expense is required.

5. CONCLUSIONS

The modelling and the formulation of governing dynamic equations for cracked rotor systems have been studied. In particular, a distributed parameter rotor substructure with uniform mass and stiffness properties and a single crack has been considered. An analytical approach has been proposed to generate model equations which can be used as a subset within an extensive system of equations of motion for a complex multi-shaft, multi-bearing rotating machine. In this way, the investigation can be focussed on the crack effect without being disturbed by some other secondary influences. Starting with a two-field formulation and transition conditions, the procedure described reduces the problem to equations for one uniform segment with a modified load distribution. Applying the classical Galerkin method, for instance, to generate ordinary differential equations, global shape functions can be used so that a halved number of governing equations of motion appears. Finally, the procedure has been illustrated discussing the torsional vibrations of a circular shaft with a circumferential crack. Even this simple example points out that the proposed approach is a powerful instrument to obtain approximate results by a relatively small calculation expense.

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REFERENCES

Bernasconi. O. (1986). Solution for torsional vibrations of stepped shafts using singularity functions. *Int. J. Mech. Sci.* **28**, 31–39.

Cawley, P. and Adams, R. D. (1979). The location of defects in structures from measurements of natural frequencies. *J. Strain Analysis* **14**, 49–57.

Chang, H.-Y. and Petroski, H. J. (1986). On detecting a crack by tapping a beam. *Int. J. Pres. Ves. Piping* **22**, 41–51.

Davies, W. G. R. and Mayes, I. W. (1984). The vibrational behavior of a multi-shaft, multi-bearing system in the presence of a propagating transverse crack. *J. Vibr. Acoust. Stress Rel. Design* **106**, 146–153.

Diana, G., Bachschmid, N. and Angeli, F. (1986). An on-line crack detection method for turbogenerator rotors. In *Proc. Int. Conf. on Rotordynamics*, p. 385, Tokyo.

Dimarogonas, A. and Massouros, G. (1981). Torsional vibrations of a shaft with a circumferential crack. *Eng. Fracture Mech.* **15**, 439–444.

Eshleman, R. L. and Eubanks, R. A. (1969). On the critical speeds of a continuous rotor. *J. Engng Industry* **91**, 1180–1187.

Gasch, R. (1976). Dynamic behavior of a simple rotor with a cross-sectional crack. In *Vibrations in Rotating Machinery*, p. 123. The Institution of Mechanical Engineers, Suffolk.

Grabowski, B. (1980). The vibrational behavior of a turbine rotor containing a transverse crack. *J. Mech. Design* **102**, 140–146.

Gudmundson, P. (1983). The dynamic behavior of slender structures with cross-sectional cracks. *J. Mech. Phys. Solids* **31**, 329–345.

Haisty, B. S., Duerr, R. E and Springer, W. T. (1988). Cracked beam modification element, part I. In *Proc. 6th Int. Modal Analysis Conf.*, Vol. 2, p. 1343, Kissimmee, Florida.

Henry, T. A. and Okah-Avae, B. E. (1976). Vibrations in cracked shafts. In *Vibrations of Rotating Machinery*, p. 15. The Institution of Mechanical Engineers, Suffolk.

Irwin, G. R. (1960). Fracture mechanics. In *Structural Mechanics* (Edited by J. N. Goodier and N. J. Hoff), p. 557. Pergamon Press, Oxford, New York.

Kelkel, K. (1978). *Auswuchten elastischer Rotoren in isotrop federnder Lagerung*. Hochschulverlag, Stuttgart.

Kirmser, P. G. (1944). The effect of discontinuities on the natural frequencies of beams. *Proc. ASTM* **44**, 887–904.

Meyer, J. (1979). Zum Schwingungsverhalten einer Laval-Welle mit angerissenem Wellenquerschnitt. Dr-Ing. Thesis, HSBW Hamburg.

Muszynska, A. (1982). Shaft crack detection. In *Proc. 7th Machinery Dynamics Seminar*, Edmonton, Canada.

Nelson, H. D. and Nataraj, C. (1986). The dynamics of a rotor system with a cracked shaft. *J. Vibr. Acoust. Stress Rel. in Design* **108,** 189–196.

Papadopoulos, C. A. and Dimarogonas, A. D. (1987a). Coupled longitudinal and bending vibrations of a rotating shaft with an open crack. *J. Sound Vibr.*, **117,** 81–93.

Papadopoulos, C. A. and Dimarogonas, A. D. (1987b). Coupling of bending and torsional vibrations of a cracked Timoshenko beam. *Ing.-Arch.* **57,** 257.

Papadopoulos, C. A. and Dimarogonas, A. D. (1988). Coupled longitudinal and bending vibrations of a cracked shaft. *J. Vibr. Acoust. Stress Rel. in Design* **110,** 1.

Pedersen, P. T. (1972). On forward and backward precession of rotors. *Ing.-Arch.* **42,** 26–41.

Petroski, H. J. (1981). Simple static and dynamic models for the cracked elastic beam. *Int. J. Fracture* **17,** R71–76.

Schmalhorst, B. K. (1988). Numerical solution of cracked rotors's vibrations due to measured crack shapes. In *Proc. 2nd Int. Symp. on Transport Phenomena, Dynamics, and Design of Rotating Machinery*. Vol. 2, p. 271, Honolulu.

Thomson, W. T. (1949). Vibration of slender bars with discontinuities in stiffness. *J. Appl. Mech.* **16,** 203–207.

Wauer, J. (1983). A general linear approach to symmetric distributed parameter rotor systems. In *Proc. 6th IFTOMM-World Congress*, Vol. 2, p. 1313, New Delhi.

Wauer, J. (1985). Stabilität viskoelastischer Wellen unter axialem Druck. *ZAMM* **65,** 379.

Wauer, J. (1986). Secondary effects in transient vibrations of rotating shafts through critical speeds. In *Proc. Int. Conf. on Vibration Problems in Engineering*, p. 785, X'ian, P.R. China.

Wauer, J. (1990). On the dynamics of cracked rotors—a literature survey. *Appl. Mech. Rev.*, to appear.